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REDUCTION OF OBSERVATION EQUATIONS WHICH CONTAIN MORE THAN ONE OBSERVED QUANTITY.

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THE ordinary case, in the application of the method of Least Squares, where a quantity is observed which is a known function of certain unknown quantities, and the case, occurring principally in higher geodesy, where the observed quantities have to fulfill rigorous conditions, have received by Gauss, Bessel, Hansen, Encke and others such an exhaustive treatment, that it would be difficult to add anything of importance to their labors. The case mentioned in the heading of this article, which is nothing but an extension of the ordinary case, whenever it occurs, is generally forced into the treatment of the ordinary case, either by considering one class of observed quantities without error (comparing standards of length at different temperatures, we really have two observed quantities, viz., difference of length and temperature, but in this case the temperature is considered in the reduction without error), or some function of the observed quantities is treated as an observed quantity.—In Mr. Hill's reduction of Regnaults' experiments on the volume of atmospheric air, ANALYST, Vol. IV, No. 4, there are three classes of observed quantities, viz., Vol., V, Pressure, P and temperature, T. In the first class of experiments, where the temperature is nearly the same there are then only two variable observed quantities, V and P, and in applying Least Squares Mr. Hill makes the sum of the squares of the corrections to $\log(PV)$ a minimum.—Upon close examination it will be found that a great many observations are of a more or less complex nature so that the observation equations contain more than one observed quantity, and it may be of interest to show how such observation equations should be treated.

Suppose we have the following m observation equations:

$$f_1(a, b, \ldots x_1+\delta x_1, y_1+\delta y_1, \ldots) = 0, \tag{1_1}$$

$$f_2(a, b, \ldots x_2+\delta x_2, y_2+\delta y_2, \ldots) = 0, \tag{1_2}$$

. . . .

$$f_m(a, b, \ldots x_m+\delta x_m, y_m+\delta y_m, \ldots) = 0, \tag{1_m}$$

where the form of these functions may be considered different, provided the same unknown quantities, a, b, ... either all or in part, occur in them. Also the observed quantities need not be of the same class in all observation equations but I suppose here each observed value to enter only one of them.

Since the corrections δx_1, $\delta y_1, \ldots$ δx_2, $\delta y_2, \ldots$ must be small quantities, we have, if $f_1, f_2, \ldots f_m$ denote the values of (1_1), (1_2), . . . (1_m), if the uncorrected values, x_1, $y_1, \ldots$ x_2, $y_2, \ldots$ x_m, $y_m, \ldots$ are substituted,

$$f_1+\frac{df_1}{dx_1}\delta x_1+\frac{df_1}{dy_1}\delta y_1+\ldots = 0, \tag{$1'_1$}$$

$$f_2+\frac{df_2}{dx_2}\delta x_2+\frac{df_2}{dy_2}\delta y_2+\ldots = 0, \tag{$1'_2$}$$

. . . .

$$f_m+\frac{df_m}{dx_m}\delta x_m+\frac{df_m}{dy_m}\delta y_m+\ldots = 0. \tag{$1'_m$}$$

Let p_1 = weight of x_1 ; q_1 = weight of y_1 ; . . . (2_1)

p_2 = " " x_2 ; q_2 = " " y_2 ; . . . (2_2)

. . . .

p_m = " " x_m ; q_m = " " y_m ; . . . (2_m)

The quantities $f_1, f_2, \ldots . f_m$ are independent with respect to the observations as we here suppose. Let P_1, P_2, P_3, ... P_m be their weights respectively then we have, evidently,

$$\frac{1}{P_1}=\frac{1}{p_1}\left(\frac{df_1}{dx_1}\right)^2+\frac{1}{q_1}\left(\frac{df_1}{dy_1}\right)^2+\ldots \tag{3_1}$$

$$\frac{1}{P_2}=\frac{1}{p_2}\left(\frac{df_2}{dx_2}\right)^2+\frac{1}{q_2}\left(\frac{df_2}{dy_2}\right)^2+\ldots \tag{3_2}$$

. . . .

$$\frac{1}{P_m}=\frac{1}{p_m}\left(\frac{df_m}{dx_m}\right)^2+\frac{1}{q_m}\left(\frac{df_m}{dy_m}\right)^2+\ldots \tag{3_m}$$

which expressions are in general functions of a, b, . . . and of the observed quantities.

According to the principle of least squares we must make

$$[p\delta x^2]+[q\delta y^2]+\ldots = \text{a minimum}, \tag{4}$$

or, by (1′) $[p\delta x^2]+\left[q\left(\frac{f+(df\div dx)\delta x+\,..\,.}{df\div dy}\right)^2\right]+\ldots=$ a min. (4′)

Equating to zero the differential coefficients with respect to δx_1, $\delta x_2, \ldots$ δx_m, successively, solving for these corrections, and doing the same for δy_1, $\delta y_2, \ldots$ δy_m, etc., we obtain regarding (3)

$$\delta x_1=-\frac{P_1 f_1}{p_1}\cdot\frac{df_1}{dx_1};\ \delta y_1=-\frac{P_1 f_1}{q_1}\cdot\frac{df_1}{dy_1}\ldots\ldots \qquad (5_1)$$

$$\delta x_2=-\frac{P_2 f_2}{p_2}\cdot\frac{df_2}{dx_2};\ \delta y_2=-\frac{P_2 f_2}{q_2}\cdot\frac{df_2}{dy_2}\ldots\ldots \qquad (5_2)$$

. . . .

$$\delta x_m=-\frac{P_m f_m}{p_m}\cdot\frac{df_m}{dx_m};\ \delta y_m=-\frac{P_m f_m}{q_m}\cdot\frac{df_m}{dy_m}\ldots\ldots \qquad (5_m)$$

and substituting these expressions into (4) we obtain

$$[Pf^2]=\text{a minimum}; \qquad (4'')$$

$$\therefore \qquad \frac{d}{2da}\left[Pf^2\right]=0, \qquad (6_a)$$

$$\frac{d}{2db}\left[Pf^2\right]=0, \qquad (6_b)$$

. . . .

Only in the simplest case these conditions may be employed for the direct solution of our problem. Before attempting the general problem, I take such a simple case. Let

$$f_1=ax_1-y_1+b=-a\delta x_1+\delta y_1, \qquad (a_1)$$

$$f_2=ax_2-y_2+b=-a\delta x_2+\delta y_2, \qquad (a_2)$$

. . . .

$$f_m=ax_m-y_m+b=-a\delta x_m+\delta y_m, \qquad (a_m)$$

be the observation equations, then we have by (3)

$$\frac{1}{P_1}=\frac{a^2}{p_1}+\frac{1}{q_1}, \qquad (b_1)$$

$$\frac{1}{P_2}=\frac{a^2}{p_2}+\frac{1}{q_2}, \qquad (b_2)$$

. . . .

$$\frac{1}{P_m}=\frac{a^2}{p_m}+\frac{1}{q_m}. \qquad (b_m)$$

We have then by (4″) $\left[\dfrac{(ax-y+b)^2}{\dfrac{a^2}{p}+\dfrac{1}{q}}\right]=$ a minimum,*

*If $p=q=1$ we have $[(ax-y+b)^2\div(a^2+1)]=$ a minimum. This is Mr. Adcock's principle of making the normals as short as possible. See ANALYST, Vol. V, p. 54.

and by (6)

$$\frac{d}{2da}\left[\frac{(ax-y+b)^2}{\frac{a^2}{p}+\frac{1}{q}}\right]=0, \qquad (c_a)$$

$$\frac{d}{2db}\left[\frac{(ax-y+b)^2}{\frac{a^2}{p}+\frac{1}{q}}\right]=0, \qquad (c_b)$$

or

$$\left[\frac{x}{\frac{a^2}{p}+\frac{1}{q}}\left(ax-y+b\right)\right]-\left[\frac{a\div p}{\left(\frac{a^2}{p}+\frac{1}{q}\right)^2}\left(ax-y+b\right)^2\right]=0, \qquad (d_a)$$

$$\left[\frac{ax-y+b}{\frac{a^2}{p}+\frac{1}{q}}\right]=0. \qquad (d_b)$$

Unless we make a certain assumption with regard to the weights of the observed quantities, we cannot solve, by a direct process, even these eq'ns. Such an assumption is: that the precisions of x and y which enter the same observation equation, are considered to have a constant ratio. The x and y observations having been made as nearly as possible simultaneously, they are made under the same conditions and are therefore as good as those conditions permit. From this however it does not follow that they have equal weight, more particularly if they are heterogeneous quantities, as for example, temperature readings and microscope readings on the comparator in comparing standards of length, or signalling the transit of a star over a thread in a telescope on the chronograph and reading the level, in time observations, etc. If weight is defined as the reciprocal square of the probable error, it is clear that the unit in which a quantity is expressed makes a great difference in the weight to be assigned. All we can safely assume therefore with regard to the weights of simultaneous observations that enter one observation equation is this: that there is, throughout, a constant ratio in their precisions, the square of which is the constant ratio of their weights. To obtain this we may take the reciprocal ratio of the practical limits of error in each class of observations for the ratio of the precisions. Such limits are known sufficiently near by any experienced observer. Even a computer can draw some inference from the results given to him, which will enable him to get some rough idea of this ratio; for a good observer will not give his results with any more accuracy than he thinks he can estimate. Now suppose x to be given to μ, and y to ν decimal places, whatever be the units in which they are expressed; then it is the computer's duty not to alter the respective *figures* (not the absolute values) any more than necessary to establish consistency. This he will effect if he takes y $10^{\nu-\mu}$ times as precise as x. If however, for example, y is the mean of k

directly observed values, he should take y $10^{\nu-\mu}\sqrt{k}$ times as precise as x, and similarly he might reason in other more complex cases. Suppose then

$$q_1 = kp_1, \qquad (e_1)$$

$$q_2 = kp_2, \qquad (e_2)$$

$$\cdot \quad \cdot \quad \cdot \quad \cdot$$

$$q_m = kp_m, \qquad (e_m)$$

then we have, substituting in (d_a) and (d_b),

$$a[px^2]-[pxy]+b[px]-\frac{a}{a^2+(1\div k)}\left\{[py^2]-2a[pxy]+a^2[px^2]-2b[py]+2ab[px]+b^2[p]\right\}=0, \qquad (d'_a)$$

$$a[px]-[py]+b[p]=0. \qquad (d'_b)$$

From (d'_b) we have $b=(1\div[p])\{[py]-a[px]\}$ (f)

Substituting this into (d'_a), we obtain, after reduction,

$$a^2+a\left\{\frac{(1\div k)([px^2]-(1\div[p])[px]^2)-([py^2]-(1\div[p])[py]^2)}{[pxy]-(1\div[p])[px][py]}\right\}-\frac{1}{k}=0, \qquad (g)$$

and solving we obtain

$$a=-\frac{(1\div k)[px^2.1]-[py^2.1]}{2[pxy.1]}\pm\sqrt{\left[\frac{1}{k}+\frac{\{(1\div k)[px^2.1]-[py^2.1]\}^2}{4[pxy.1]^2}\right]}, \qquad (h)$$

where use has been made of Gauss' notation of auxiliaries as used in his method of substitution.

To decide which of the signs must be taken for the minimum minimorum, I take two extreme cases:

1. Let the weight of the x be infinite, or let us suppose the x without error, then $k=0$ in order that $q=kp=$ a finite quantity. Then (h) becomes

$$a'=\frac{[qxy.1]}{[qx^2.1]} \text{ or } =\infty. \qquad (h')$$

2. Let the y be without error, or $k=\infty$, then (h) becomes

$$a''=\frac{[py^2.1]}{[pxy.1]} \text{ or } =0. \qquad (h'')$$

Only the results given first, which correspond to the upper sign, can be used; for they agree with the results from the ordinary treatment if 1, the (y)'s and 2, the (x)'s are corrected. We have then to adopt the upper sign in (h).

If $p_1=p_2=\dots p_m=1$ and also $k=1$ we have Mr. Adcock's problem, Vol. V, page 54, and (h) becomes

$$a=-\frac{[x^2]-(1\div m)[x]^2-[y^2]+(1\div m)[y]^2}{2[xy]-(2\div m)[x][y]}$$

$$+\sqrt{1+\frac{1}{4}\left\{\frac{[x^2]-(1\div m)[x]^2-[y^2]+(1\div m)[y]^2}{[xy]-(1\div m)[x][y]}\right\}^2}. \quad (h_0)$$

This however is not identical with Mr. Adcock's formula in consequence of an error which he committed in eliminating b.

Having found a and b we compute the f and P by formulæ (b) and (c), and the minimum sum of squares by (4″).

We have then the probable error of an observation of weight 1, since there are two unknown quantities,

$$r=\rho\sqrt{\left(\frac{2[pf^2]}{m-2}\right)}. \quad (i)$$

Although the probable errors of a and b could be deduced in terms of the probable error of observation from their expressions just given, yet since this requires tedious operations and since the approximate solution which I am going to give of the general problem also gives the probable errors of the unknown quantities, I pursue this special case no farther.

Assume

$$P_1=\frac{1^*}{N_1}\dots(7_1);\ P_2=\frac{1}{N_2}\dots(7_2);\dots P_m=\frac{1}{N_m}; \quad (7_m)$$

then (4″) becomes $\quad [f^2\div N]=$ a minimum. $\quad$ (4‴)

Let $a_0, b_0, \dots$ be some approximate values of the unknown quantities, and $\Delta a_0, \Delta b_0, \dots$ their corrections, then we have, in (4‴) exact to quantities of the second order,

$$\begin{aligned}\left[\frac{f^2}{N}\right]=\left[\frac{f^2}{N}\right]_0&+\frac{d}{da}\left[\frac{f^2}{N}\right]_0\Delta a_0+\frac{d}{db}\left[\frac{f^2}{N}\right]_0\Delta b_0+\dots\\&+\frac{d^2}{2da^2}\left[\frac{f^2}{N}\right]_0\Delta a_0^2+\frac{d^2}{2dadb}\left[\frac{f^2}{N}\right]_0\Delta a_0\Delta b_0+..\\&+\frac{d^2}{2dadb}\left[\frac{f^2}{N}\right]_0\Delta a_0\Delta b_0+\frac{d^2}{2db^2}\left[\frac{f^2}{N}\right]_0\Delta b_0^2+\dots\\&+\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ .\ . \qquad (4'''')\\&=\text{a minimum};\end{aligned}$$

and the conditions for a minimum, or the normal equations, are

$$\frac{d}{2\Delta a_0}\left[\frac{f^2}{N}\right]=\frac{d}{2da}\left[\frac{f^2}{N}\right]_0+\frac{d^2}{2da^2}\left[\frac{f^2}{N}\right]_0\Delta a_0+\frac{d^2}{2dadb}\left[\frac{f^2}{N}\right]_0\Delta b_0+\dots=0, \quad (6'_a)$$

$$\frac{d}{2\Delta b_0}\left[\frac{f^2}{N}\right]=\frac{d}{2db}\left[\frac{f^2}{N}\right]_0+\frac{d^2}{2dadb}\left[\frac{f^2}{N}\right]_0\Delta a_0+\frac{d^2}{2db^2}\left[\frac{f^2}{N}\right]_0\Delta b_0+\dots=0, \quad (6'_b)$$

. . . .

*This is done because the N's are easier to differentiate than the P's.

We have

$$\frac{d}{2da}\left[\frac{f^2}{N}\right]_0 = \left[\frac{f}{N}\left(\frac{df}{da}-\frac{f}{2N}\cdot\frac{dN}{da}\right)\right]_0, \tag{8_a}$$

$$\frac{d}{2db}\left[\frac{f^2}{N}\right]_0 = \left[\frac{f}{N}\left(\frac{df}{db}-\frac{f}{2N}\cdot\frac{dN}{db}\right)\right]_0, \tag{8_b}$$

. . . .

$$\frac{d^2}{2\,da^2}\left[\frac{f^2}{N}\right]_0 = \left[\frac{1}{N}\left(\frac{df}{da}-\frac{f}{N}\cdot\frac{dN}{da}\right)^2\right]_0 + \left[\frac{f}{N}\left(\frac{d^2f}{da^2}-\frac{f}{2N}\cdot\frac{d^2N}{da^2}\right)\right]_0, \tag{8_{aa}}$$

$$\frac{d^2}{2da\,db}\left[\frac{f^2}{N}\right]_0 = \left[\frac{1}{N}\left(\frac{df}{da}-\frac{f}{N}\cdot\frac{dN}{da}\right)\left(\frac{df}{db}-\frac{f}{N}\cdot\frac{dN}{db}\right)\right]_0$$

$$+\left[\frac{f}{N}\left(\frac{d^2f}{da\,db}-\frac{f}{2N}\cdot\frac{d^2N}{da\,db}\right)\right]_0, \tag{8_{ab}}$$

. . . .

$$\frac{d^2}{2\,db^2}\left[\frac{f^2}{N}\right]_0 = \left[\frac{1}{N}\left(\frac{df}{db}-\frac{f}{N}\cdot\frac{dN}{db}\right)^2\right]_0 + \left[\frac{f}{N}\left(\frac{d^2f}{db^2}-\frac{f}{2N}\cdot\frac{d^2N}{db^2}\right)\right]_0, \tag{8_{bb}}$$

. . . .

Having computed the first and second differential coefficient of f and N, these coefficients can be found and used for the normal equations (6′).

If we place

$$\frac{df_1}{da}-\frac{f_1}{2N_1}\cdot\frac{dN_1}{da} = X_1;\quad \frac{df_1}{db}-\frac{f_1}{2N_1}\cdot\frac{dN_1}{db} = Y_1, \ldots \tag{9_1}$$

$$\frac{df_2}{da}-\frac{f_2}{2N_2}\cdot\frac{dN_2}{da} = X_2;\quad \frac{df_2}{db}-\frac{f_2}{2N_2}\cdot\frac{dN_2}{db} = Y_2, \ldots \tag{9_2}$$

. . . .

$$\frac{df_m}{da}-\frac{f_m}{2N_m}\cdot\frac{dN_m}{da} = X_m;\quad \frac{df_m}{db}-\frac{f_m}{2N_m}\cdot\frac{dN_m}{db} = Y_m, \ldots \tag{9_m}$$

we have sufficiently near (replacing now 1÷N by P)

$$[PXf]_0+[PX^2]_0\Delta a_0+[PXY]_0\Delta b_0+\ldots = 0, \tag{$6''_a$}$$

$$[PYf]_0+[PXY]_0\Delta a_0+[PY^2]_0\Delta b_0+\ldots = 0, \tag{$6''_b$}$$

. . . .

These normal equations we might consider to have been formed from the following fictitious observation equations:

$$\Delta_1 = f_1+X_1\Delta a_0+Y_1\Delta b_0+\ldots\ldots (\text{weight} = P_1) \tag{10_1}$$

$$\Delta_2 = f_2+X_2\Delta a_0+Y_2\Delta b_0+\ldots\ldots (\text{weight} = P_2) \tag{10_2}$$

. . . .

$$\Delta_m = f_m+X_m\Delta a_0+Y_m\Delta b_0+\ldots\ldots (\text{weight} = P_m) \tag{10_m}$$

if $[P\Delta^2]$ is made a minimum.

The equations $(6''_a)$, $(6''_b)$ . . . not only give the corrections $\varDelta a_0$, $\varDelta b_0$, . . . to the assumed values of the unknown quantities, but also their weights in the usual manner.

If n = number of unknown quantities there are $m-n$ conditions between the observed quantities alone. We have then for the probable error of an observation of weight 1

$$r = \rho\sqrt{\left(\frac{2[Pf^2]}{m-n}\right)}; \tag{11}$$

where $[Pf^2]$ is to be computed by means of the final values of a, b, . . . and if p_a, p_b, . . . are the weights of a, b, . . . as deduced from $(6''_a)$, $(6''_b)$ . . . we have their probable errors,

$$r_a = \frac{r}{\sqrt{p_a}}, \tag{12_a}$$

$$r_b = \frac{r}{\sqrt{p_b}}, \tag{12_b}$$

. . . . .

Applying this process to the above simple case we have

$$\frac{df_1}{da} = x_1\,; \quad \frac{df_1}{db} = 1\,; \quad \frac{d^2 f_1}{da^2} = \frac{d^2 f_1}{dadb} = \frac{d^2 f_1}{db^2} = 0, \tag{j_1}$$

$$\frac{df_2}{da} = x_2\,; \quad \frac{df_2}{db} = 1\,; \quad \frac{d^2 f_2}{da^2} = \frac{d^2 f_2}{dadb} = \frac{d^2 f_2}{db^2} = 0, \tag{j_2}$$

. . . .

$$\frac{d\mathrm{N}_1}{da} = \frac{2a}{p_1}; \quad \frac{d\mathrm{N}_1}{db} = 0\,; \quad \frac{d^2\mathrm{N}_1}{da^2} = \frac{2}{p_1}; \quad \frac{d^2\mathrm{N}_1}{dadb} = \frac{d^2\mathrm{N}_1}{db^2} = 0, \tag{k_1}$$

$$\frac{d\mathrm{N}_2}{da} = \frac{2a}{p_2}; \quad \frac{d\mathrm{N}_2}{db} = 0\,; \quad \frac{d^2\mathrm{N}_2}{da^2} = \frac{2}{p_2}; \quad \frac{d^2\mathrm{N}_2}{dadb} = \frac{d^2\mathrm{N}_2}{db^2} = 0, \tag{k_2}$$

. . . .

We have then

$$\frac{d}{2da}\left[\frac{f^2}{\mathrm{N}}\right]_0 = \left[\frac{f}{\mathrm{N}}\left(x - \frac{f}{\mathrm{N}}\frac{a}{p_1}\right)\right]_0 = \left[Pf\left(x - \frac{Pf}{p}a\right)\right]_0 \tag{l_a}$$

$$\frac{d}{2db}\left[\frac{f^2}{\mathrm{N}}\right]_0 = \left[\frac{f}{\mathrm{N}}\right]_0 \qquad = \left[Pf\right]_0, \tag{l_b}$$

$$\frac{d^2}{2da^2}\left[\frac{f^2}{\mathrm{N}}\right]_0 = \left[P\left(x - \frac{2Pf}{p}a\right)^2\right]_0 - \left[\frac{P^2}{p}f^2\right]_0, \tag{l_{aa}}$$

$$\frac{d^2}{2dadb}\left[\frac{f^2}{\mathrm{N}}\right]_0 = \left[P\left(x - \frac{2Pf}{p}a\right)\right]_0, \tag{l_{ab}}$$

$$\frac{d^2}{2db^2}\left[\frac{f^2}{\mathrm{N}}\right]_0 = \left[P\right]_0. \tag{l_{bb}}$$

Substituting into (6′) we have

$$\left[Pf\left(x-\frac{Pf}{p}a\right)\right]_0+\left[P\left(x-\frac{2Pf}{p}a\right)^2-\frac{P^2}{p}f^2\right]_0\Delta a_0$$
$$+\left[P\left(x-\frac{2Pf}{p}a\right)\right]_0\Delta b_0=0,\quad (m_a)$$
$$\left[Pf\right]_0+\left[P\left(x-\frac{2Pf}{p}a\right)\right]_0\Delta a_0+\left[P\right]_0\Delta b_0=0.\quad (m_b)$$

Employing (9) (6″) and (10) we have also

$$X_1=x_1-\frac{P_1f_1}{p_1}a\,;\quad Y_1=1,\qquad (n_1)$$
$$X_2=x_2-\frac{P_2f_2}{p_2}a\,;\quad Y_2=1,\qquad (n_2)$$

. . . . . .

Should the results of these two methods differ materially, the indication is that there is considerable uncertainty about the determination of the unknown quantities or that their probable errors are large. It would not be judicious in that case to attempt a more exact solution.

SYMMETRICAL FUNCTIONS OF THE SINES OF THE ANGLES INCLUDED IN THE EXPRESSION

$$\alpha_0+\frac{2k\pi}{n}.$$

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Denoting these angles by α_0, α_1, α_2, ... α_{n-1}, the values of sin $n\alpha$ and cos $n\alpha$ will be the same for all; hence, putting x for sin θ and $n\alpha$ for $n\theta$ in the developments of cos $n\theta$ and sin $n\theta$ in terms of sin θ (quoted on page 17) we have the eq'ns whose roots are sin α_0, sin α_1, etc.; viz., when n is even,

$$\cos n\alpha=1-\frac{n^2}{2}x^2+\frac{n^2(n^2-2^2)}{4!}x^4\ldots+(-1)^{\frac{1}{2}n}\frac{n^2(n^2-2^2)(n^2-4^2)\ldots[n^2-(n-2)^2]}{n!}x^n,\qquad (1)$$

and when n is odd,

$$\sin n\alpha=nx\qquad-\frac{n(n^2-1^2)}{3!}x^3\ldots+(-1)^{\frac{n-1}{2}}\frac{n(n^2-1^2)(n^2-3^2)\ldots[n^2-(n-2)^2]}{n!}x^n.\qquad (2)$$

On dividing by the coefficient of x^n each of these eq'ns takes the form

$$0=x^n-\frac{n(n-1)}{n^2-(n-2)^2}x^{n-2}+\frac{n(n-1)(n-2)(n-3)}{[n^2-(n-2)^2][n^2-(n-4)^2]}x^{n-4}-\ldots,$$

the equations differing only in the last terms. Reducing the coefficients,